Tutorial notes - 6 whizkiddeng@gmail.com.

The implicit function thm (IFT)

Remark: The IFT is not an easy topic, it contains many subtle restrictions, But for our course, we just regard it as an application of chain-rule, so you just have to know how to compute the derivatives of implicit functions.

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functions like : $y=f(x)$, we call them explicit form no matter how complicate $f(x)$ is, for at least we can separate x and y.

For such functions, derivative can be get directly like $\frac{dy}{dx}=f'(x)$, $\frac{d^2y}{dx^2}=y''=f''(x)\ldots$

But in most cases we just have implicit form, write as:

$$F(x,y)=0 \qquad (1)$$

like: $x-y-\frac{1}{2}\sin y=0$, we can't have $y=f(x)$ form.

In order to compute its derivative, we use the chain-rule:

First we assume $y=f(x)$ near the point $x=x_0$ that we want to solve derivative indeed, though we can't write out $y=f(x)$ form.

So (1) $\Rightarrow F(x,y)=0 \Rightarrow F(x,f(x))=0$.

do differential both sides:

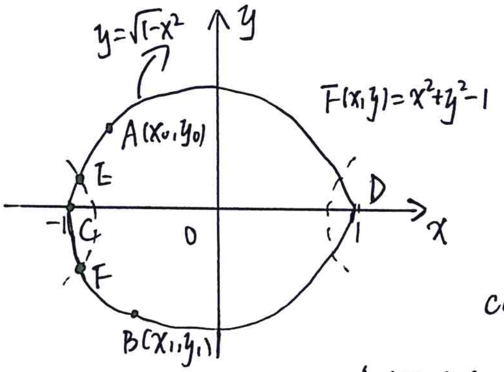
$$F_x\frac{dx}{dx}+F_y\cdot\frac{dy}{dx}=0$$

(F_x, F_y denotes partial derivatives when we regard x,y as independent variables)

$$F_x(x_0,y_0)+F_y(x_0,y_0)\left.\frac{dy}{dx}\right|_{x=x_0}=0$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=x_0}=-\frac{F_x(x_0,y_0)}{F_y(x_0,y_0)} \quad \text{if } F_y(x_0,y_0)\neq 0.$$

the restriction $F_y(x_0,y_0) \neq 0$ is important, only in ~~this~~ such points can we solve the derivatives which means we have $y=f(x)$ near $x=x_0$.



like $F(x,y)=x^2+y^2-1=0$

a unit circle. So, $\begin{cases} F_x(x,y)=2x \\ F_y(x,y)=2y. \end{cases}$

For any points (x_0,y_0), if $y_0 \neq 0$, then we can have $y=f(x)$ "locally".

$A(x_0,y_0)$ $y_0>0 \Rightarrow y=\sqrt{1-x^2}$, near $x=x_0$

$B(x_1,y_1)$ $y_1<0 \Rightarrow y=-\sqrt{1-x^2}$, near $x=x_1$.

But we have C, D bad points for $F_y(C)=F_y(D)=0$, it can be seen from no matter how small the neighbour is, the curve would have more than one cross-points for a single x, this is a contradiction.

Q1. $x-y-\frac{1}{2}\sin y=0.$

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{1}{-1-\frac{1}{2}\cos y}=\frac{2}{2+\cos y}$$ for $2+\cos y \neq 0$, the domain is R

Q2. $\ln\sqrt{x^2+y^2}=\arctan\frac{y}{x} \Rightarrow \frac{1}{2}\ln(x^2+y^2)=\arctan\frac{y}{x}$

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{\frac{1}{2}\frac{2x}{x^2+y^2}-\frac{1}{1+\frac{y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right)}{\frac{1}{2}\frac{2y}{x^2+y^2}-\frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x}}=-\frac{\frac{x+y}{x^2+y^2}}{\frac{y-x}{x^2+y^2}}=\frac{x+y}{x-y} \quad (y \neq x)$$

Q3. $x^3+y^3-3axy=0$

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{3x^2-3ay}{3y^2-3ax}=-\frac{x^2-ay}{y^2-ax} \quad (y^2 \neq ax)$$

similarly you can try to solve $\frac{dx}{dy}$ just means we have curve like $x=g(y)$.

Sketch the curve.

we try to sketch the curve by using information of 3 levels.

1. value : some special points like roots $f(x_0) = 0$.

2. first derivative : $f'(x_0) = 0$ $\quad x_0$ critical points $\begin{cases} \text{local max} \\ \text{local min} \\ \text{neither.} \end{cases}$

And $f'(x) > 0$, increasing; $f'(x) < 0$ decreasing.

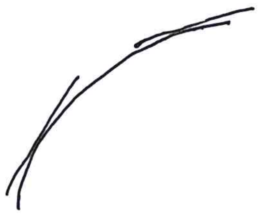
3. Second order derivative: the convex and concave properties.

when $f'(x) > 0$, $f(x) \nearrow$ we still have 2 cases.

(1) $f''(x) > 0$ which means $f'(x) \nearrow$, like:

the slope become bigger.

(2) $f''(x) < 0 \Rightarrow f'(x) \downarrow$, like:



the slope become smaller.

Q1. $f(x) = \sqrt[3]{x^3 - x^2 - x + 1} = (x-1)^{\frac{2}{3}} (x+1)^{\frac{1}{3}}$ $\quad D_f = R$.

1. $f(x) = 0 \Rightarrow x = 1$ or -1.

2. $f'(x) = \frac{1}{3} \frac{3x+1}{(x-1)^{\frac{1}{3}}(x+1)^{\frac{2}{3}}} \Rightarrow f'(x) = 0 \Rightarrow x = -\frac{1}{3}$.

consider $f'(x) > 0 \Rightarrow x > 1$ or $x < -\frac{1}{3}$, $\nearrow$

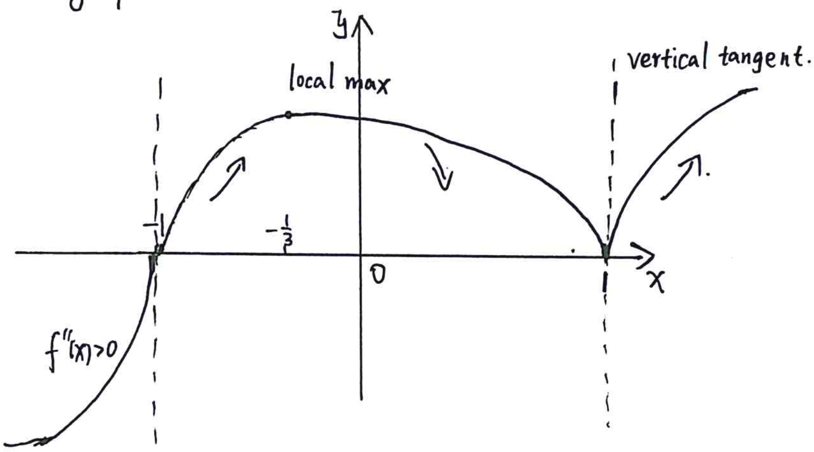
$f'(x) < 0 \Rightarrow -\frac{1}{3} < x < 1$, $\downarrow$.

and $f'(x) \to \infty$ at $x = 1, -1$ means we have vertical tangent line.

3. $f''(x) = \frac{1}{3} \cdot \frac{1}{(x-1)^{\frac{2}{3}}(x+1)^{4/3}} \cdot (-\frac{8}{3}) \cdot \frac{1}{(x-1)^{\frac{2}{3}}(x+1)^{\frac{1}{3}}}$

So we can see when $\begin{cases} x<-1, & f''(x)>0. \\ x>-1, & f''(x)<0. \end{cases}$

So the graph would be:



Tutorial 6

Topics:
- Implicit function theorem
- Introduction to implicit differentiation
- Revisit: continuity and differentiability (has been frequently asked by students)

Q1) Compute $\frac{dy}{dx}$ of the following implicit functions

a) $y^2 - x = 0$ b) $x^2 + y^2 = 1$ c) $xe^{xy} = 1$

Q2) (Revisit) Let $f : \mathbb{R} \to \mathbb{R}$; $f(x) = \begin{cases} x^2 \sin(x^{-1}) , & x \neq 0 \\ 0 , & x = 0 \end{cases}$

a) determine whether $f'(0)$ exists

b) determine whether $f'(x)$ is continuous at $x = 0$.

- <u>Implicit function theorem</u>

Let $f(x,y)$ be a continuously differentiable function.

If $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$

for some $x_0, y_0 \in \mathbb{R}$ s.t. $f(x_0, y_0) = 0$

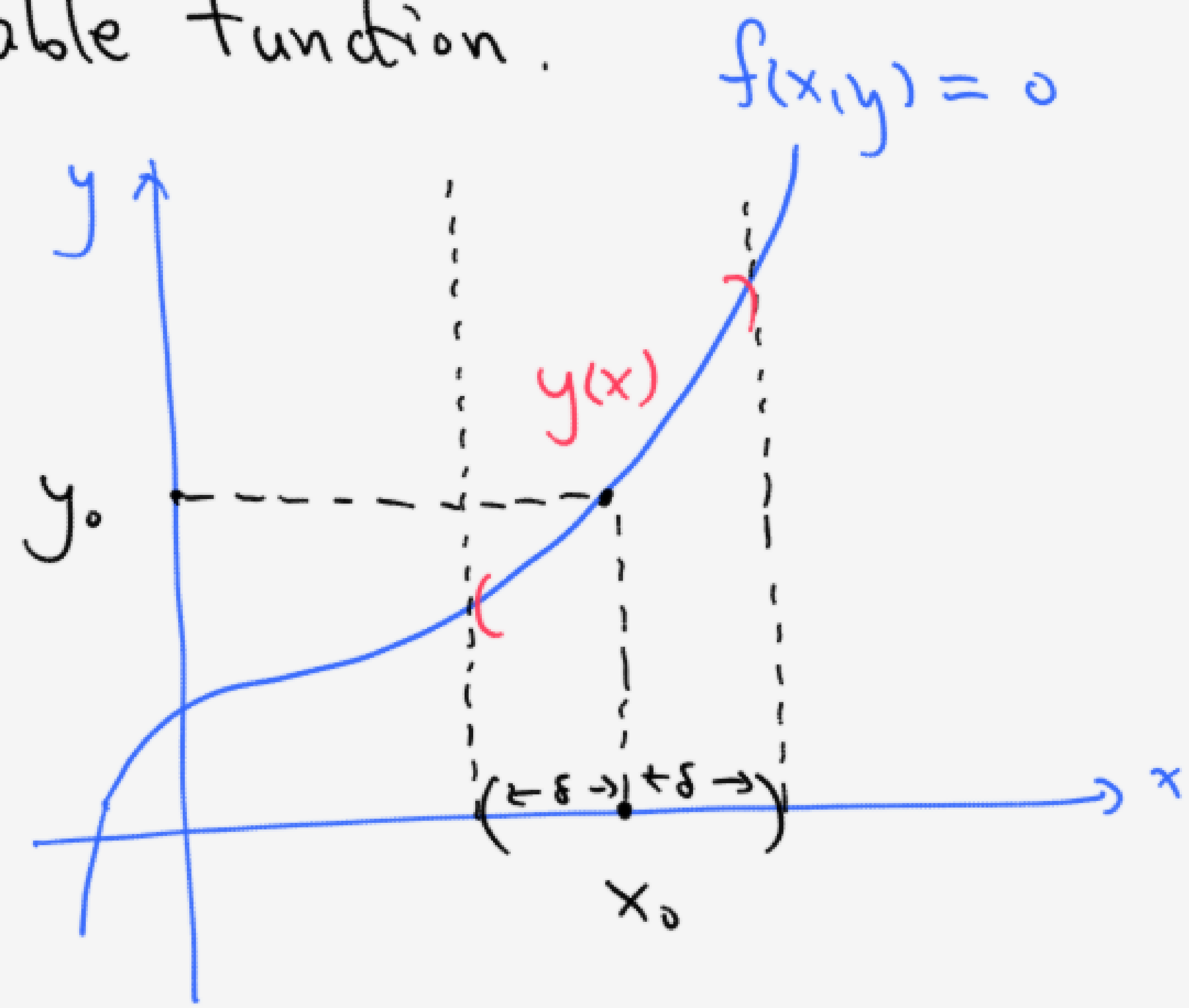
then there exist a small number $\delta > 0$

s.t. $f(x, y(x)) = 0$

$\forall\, x \in (x_0 - \delta, x_0 + \delta)$

where $y(x)$ is a continuously differentiable function

on $x \in (x_0 - \delta, x_0 + \delta)$ and $y(x_0) = y_0$



- Implicit differentiation

Let $f(x,y)$ be a continuously differentiable function

if $\frac{\partial f}{\partial y}(x_0,y_0) = 0$ for some $x_0, y_0 \in \mathbb{R}$ s.t. $f(x_0,y_0) = 0$

and $y(x)$ be the implicit function on $x \in (x_0-\delta, x_0+\delta)$

then

$$\frac{dy}{dx}(x_0) = - \frac{\frac{\partial f}{\partial x}(x_0,y_0)}{\frac{\partial f}{\partial y}(x_0,y_0)}$$

Reason: $f(x, y(x)) = 0 \quad \forall x \in (x_0-\delta, x_0+\delta)$

$$\Rightarrow \quad 0 = \left.\frac{d}{dx}\right|_{x=x_0} f(x, y(x)) = \frac{\partial f}{\partial x}(x_0,y_0) + \frac{dy}{dx}(x_0)\frac{\partial f}{\partial y}(x_0,y_0)$$

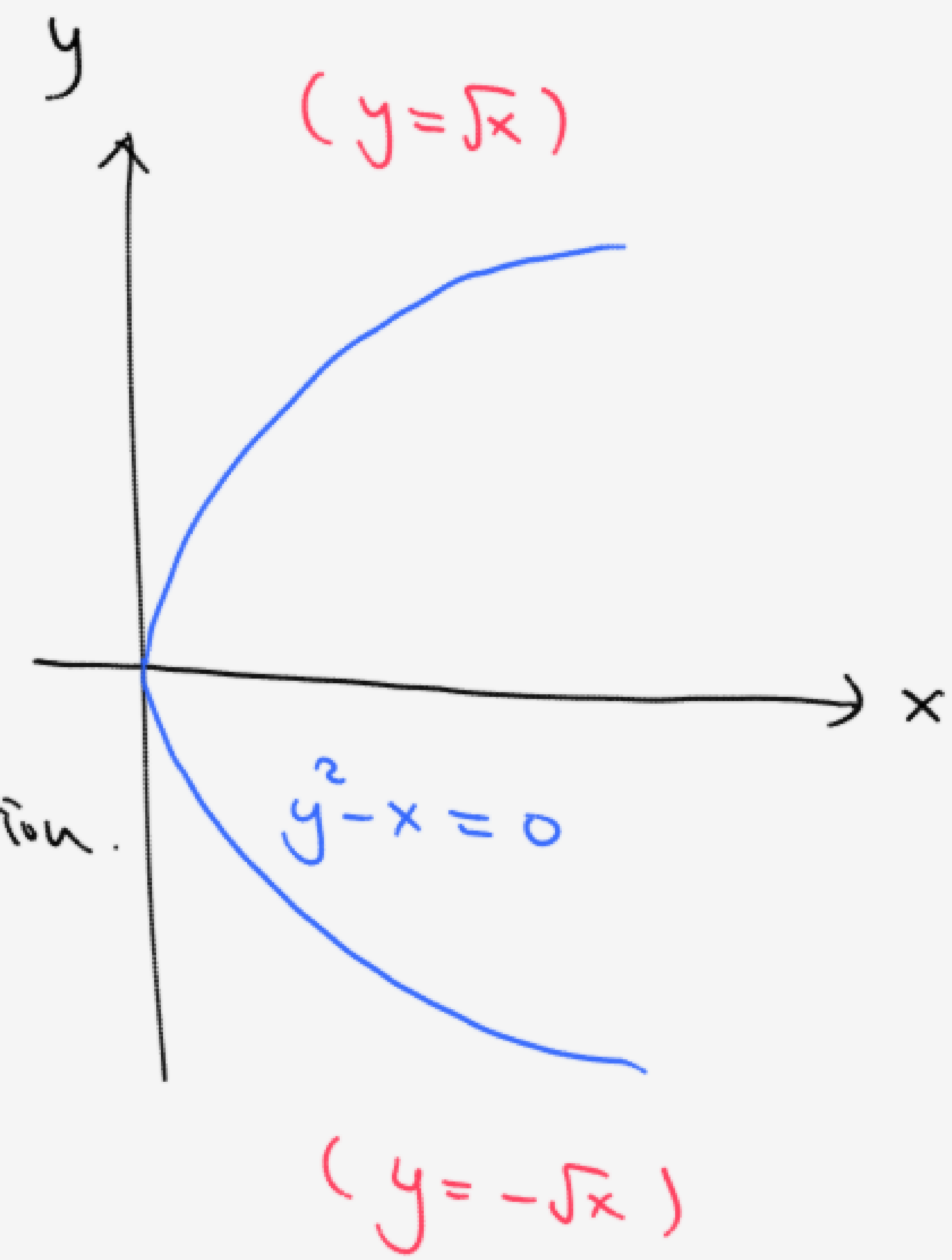
$\underline{\text{Sol}^{\underline{n}}}$

1(a) given $y^2 - x = 0$

Case 1 if $y_0 > 0$ satisfying $y_0^2 - x_0 = 0$

then $y = y(x) = \sqrt{x}$ is the implicit function.

Hence $\frac{dy}{dx}(x_0) = \frac{d}{dx}\Big|_{x=x_0} \sqrt{x} = \frac{1}{2x_0}$ //

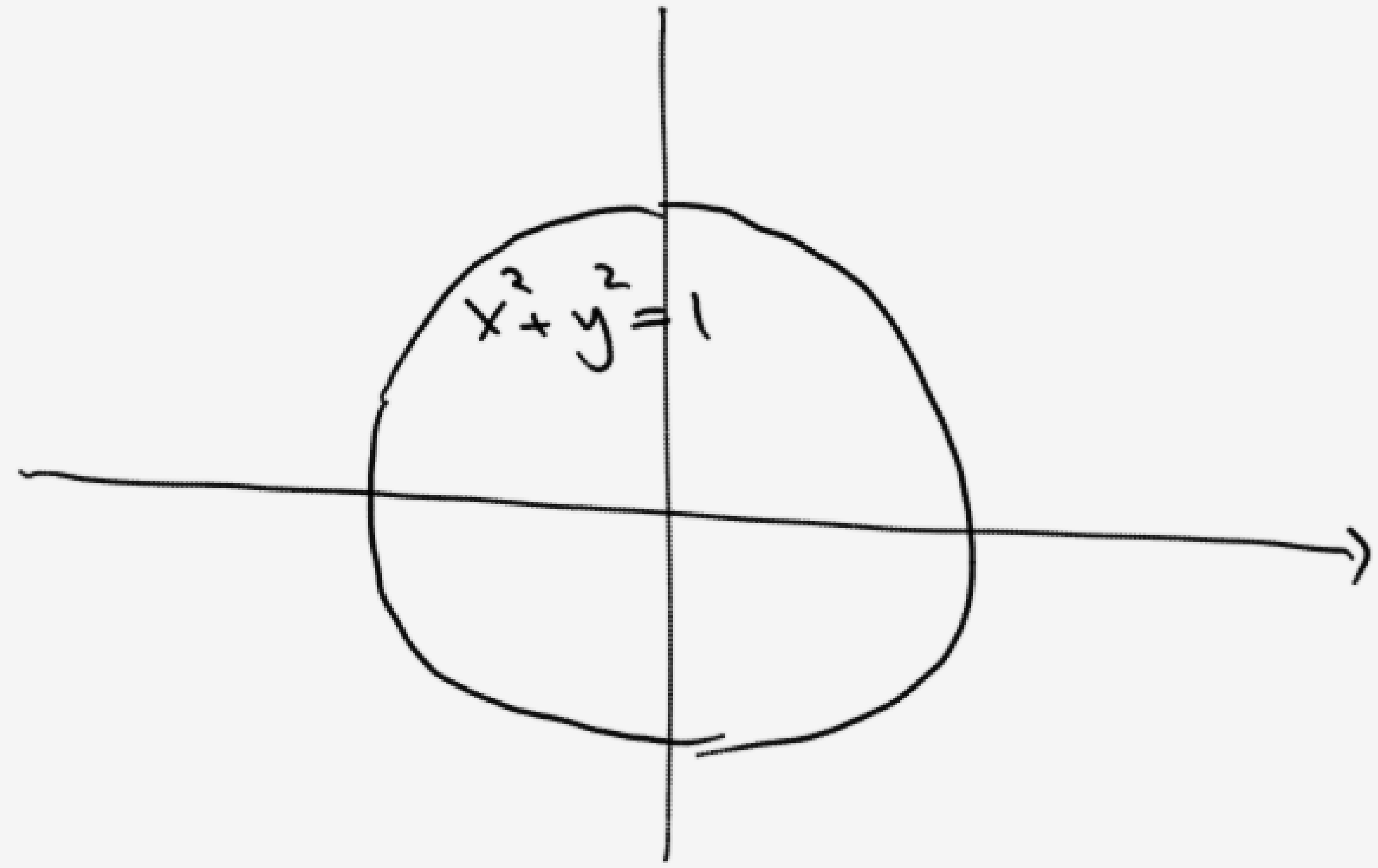


Case 2 if $y_0 < 0$ satisfying $y_0^2 - x_0 = 0$

then $y = y(x) = -\sqrt{x}$ is the implicit function

Hence $\frac{dy}{dx}(x_0) = \frac{d}{dx}\Big|_{x=x_0} -\sqrt{x} = \frac{-1}{2x_0}$ //

1b) given $x^2 + y^2 = 1$



consider $f(x,y) = x^2 + y^2 - 1$

$\frac{\partial f}{\partial y}(x,y) = 2y$,

for $y \neq 0 \Rightarrow \frac{\partial f}{\partial y}(x,y) \neq 0$

Let $x_0 \in \mathbb{R}$, $y_0 \in \mathbb{R} \setminus \{0\}$ s.t. $f(x_0, y_0) = 0$ since $\frac{\partial f}{\partial y}(x_0, y_0) = 2y_0 \neq 0$

$y(x)$ exists s.t. $y(x_0) = y_0$ for $x \in (x_0 - \delta, x_0 + \delta)$ $\exists \delta > 0$.

$0 = f(x, y(x)) = x^2 + y^2(x) - 1 \quad \forall x \in (x_0 - \delta, x_0 + \delta)$

$\Rightarrow \quad 0 = \frac{d}{dx}\Big|_{x=x_0} (x^2 + y^2(x) - 1) = 2x_0 + 2y(x_0) \frac{dy}{dx}(x_0)$

$\Rightarrow \quad \frac{dy}{dx}(x_0) = -\frac{x_0}{y_0}$

(c) given $xe^{xy} = 1$, Let $f(x,y) = xe^{xy} - 1$

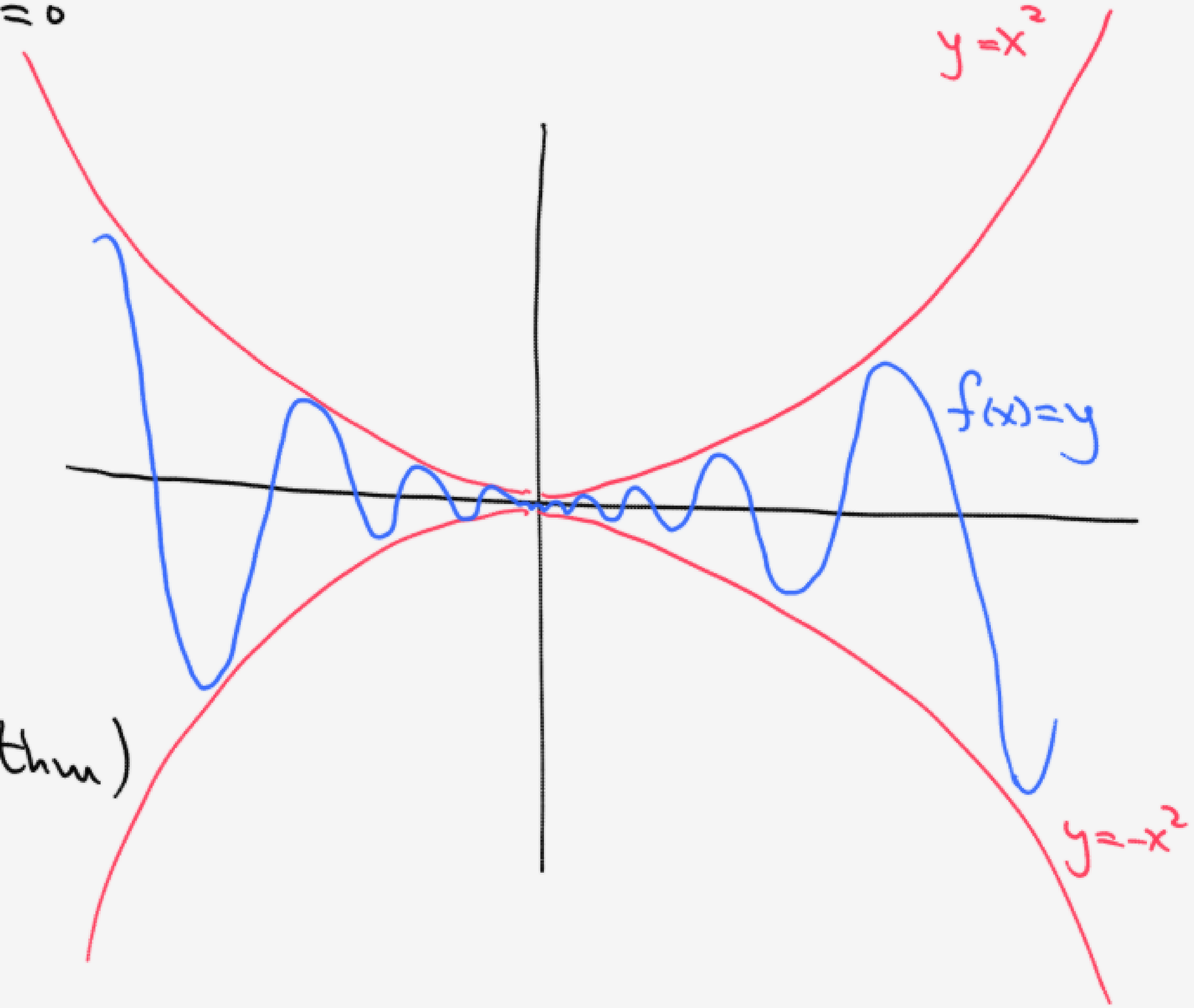
$$\frac{\partial f}{\partial y}(x,y) = x^2 e^{xy} \Rightarrow \frac{\partial f}{\partial y}(x,y) = 0 \text{ iff } x = 0$$

let $x_0 \in \mathbb{R}\setminus\{0\}$, $y_0 \in \mathbb{R}$ s.t. $f(x_0, y_0) = 0$

$$\frac{dy}{dx}(x_0,y_0) = \frac{-\frac{\partial f}{\partial x}(x_0,y_0)}{\frac{\partial f}{\partial y}(x_0,y_0)} = \frac{e^{x_0 y_0} + x_0 y_0 e^{x_0 y_0}}{x_0^2 e^{x_0 y_0}} = \frac{1 + x_0 y_0}{x_0^2}$$

//

2) given $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$



a)

$$f'_+(0) := \lim_{h \to 0^+} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0^+} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h}$$

$$= \lim_{h \to 0^+} h \sin\left(\frac{1}{h}\right) = 0 \quad \text{(by squeeze thm)}$$

$$f'_-(0) = \lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h}$$

$$= \lim_{h \to 0^-} h \sin\left(\frac{1}{h}\right) = 0 \quad \text{(again by squeeze thm)}$$

So $f'_+(0) = f'_-(0) = 0$ hence $f'(0) = 0$ exists.

2b) for $x \neq 0$, $f(x) = x^2 \sin\left(\frac{1}{x}\right)$

$$f'(x) = \frac{d}{dx}\left(x^2 \sin\left(\frac{1}{x}\right)\right) \qquad \forall x \in \mathbb{R} \setminus \{0\}$$

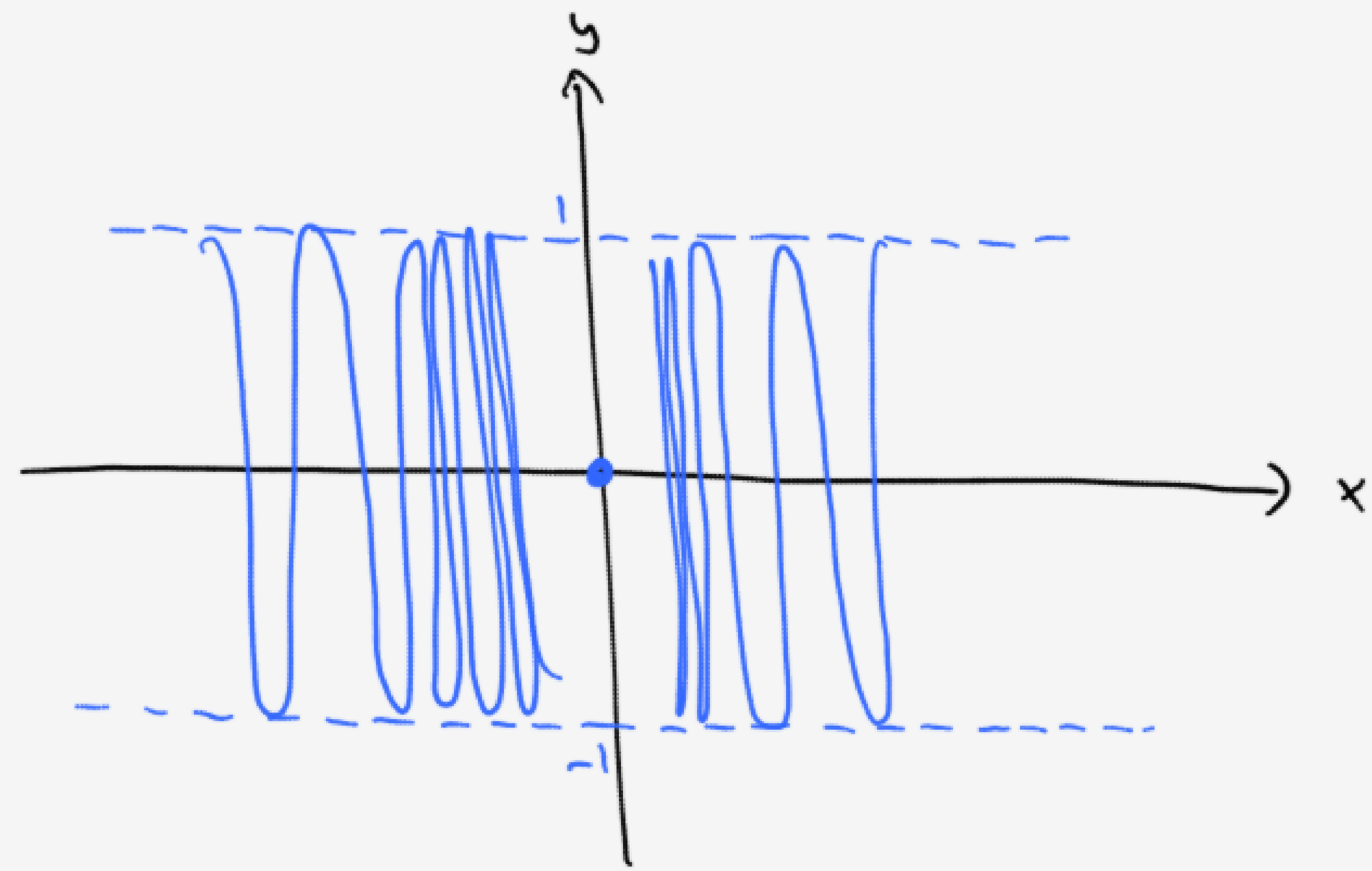
$$= 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

Notice that as $x \to 0$, $x\sin\left(\frac{1}{x}\right)$ tends to 0 by squeeze thm

BUT $\cos\left(\frac{1}{x}\right)$ oscillates between -1 and 1

Hence

$f'(x)$ is not continuous at $x=0$.



Remark: Although f' is not continuous at $x=0$, f' exists at $x=0$ and f is differentiable at $x=0$.